Univerzita Komenského v Bratislave

Fakulta matematiky, fyziky a informatiky

Mgr. Michal Daniška

Autoreferát dizertačnej práce

Study of classical and quantum phase transitions on non-Euclidean geometries in higher dimensions

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Predkladateľ:	Mgr. Michal Daniška Centrum pre výskum kvantovej informácie Fyzikálny ústav Slovenská akadémia vied Dúbravská cesta 9 845 11 Bratislava
Školiteľ:	Mgr. Andrej Gendiar, PhD. Centrum pre výskum kvantovej informácie Fyzikálny ústav SAV Bratislava
Oponenti:	

Obhajoba dizertačnej práce sa koná dňa .... o ....h pred komisiou pre obhajobu dizertačnej práce v odbore doktorandského štúdia vymenovanou predsedom odborovej komisie dňa ....

v študijnom odbore 4.1.2 všeobecná fyzika a matematická fyzika

na ....

Predseda odborovej komisie:

prof. RNDr. Peter Prešnajder, DrSc.
Katedra teoretickej fyziky a didaktiky fyziky
Fakulta matematiky, fyziky a informatiky
Univerzita Komenského
Mlynská dolina
842 48 Bratislava

Abstrakt

Zaujímavou oblasťou výskumu v modernej fyzike je vyšetrovanie správania sa klasických a kvantových systémov na neeuklidovských povrchoch v blízkosti fázového prechodu. Špecifické vlastnosti hyperbolickej geometrie však neumožňujú riešiť takéto systémy analyticky, dokonca ani štandardnými numerickými metódami. Preto otázka vhodného prístupu k analýze fermiónových modelov na hyperbolických mriežkach v termodynamickej limite zostáva nedoriešená. V prípade klasických spinových systémov bolo navrhnuté zovšeobecnenie algoritmu Corner Transfer Matrix Renormalization Group, ktoré bolo úspešne použité pri riešení spinových modelov na nekonečne veľkom počte pravidelných hyperbolických mriežok. V tejto práci rozširujeme uvedený algoritmus na špeciálne typy trojuholníkových mriežok. Ukážeme, že hyperbolická geometria indukuje u všetkých spinových modelov v bodoch fázového prechodu správanie zodpovedajúce triede univerzality stredného poľa. Je tiež dôležité zdôrazniť, že doposiaľ ešte neboli vytvorené vhodné numerické algoritmy pre štúdium kvantových systémov v základnom stave v podobných podmienkach. V tejto práci preto ponúkame jedno konkrétne riešenie tohto problému reprezentované návrhom variačného numerického algoritmu Tensor Product Variational Formulation, ktorý aproximuje kvantový základný stav v tvare súčinu nízkorozmerných uniformných tenzorov. Metódu Tensor Product Variational Formulation využívame pri štúdiu troch základných kvantových modelov na viacerých pravidelných hyperbolických mriežkach. Rovnako, ako tomu bolo v prípade klasických spinových systémov, bez ohľadu na model pozorujeme v okolí fázového prechodu správanie zodpovedajúce triede univerzality stredného poľa. Hlavné výstupy tejto práce možno zaradiť do nasledujúcich troch oblastí: (1) Navrhli sme algoritmus na výpočet a klasifikáciu termodynamických vlastností Isingovho modelu na trojuholníkovej hyperbolickej mriežke. Ďalej vyšetrujeme pôvod správania zodpovedajúceho triede univerzality stredného poľa na mriežkach s malou negatívnou krivosťou. (2) Vytvorili sme algoritmus Tensor Product Variational Formulation, pomocou ktorého numericky analyzujeme základný stav kvantových systémov na hyperbolicky zakrivených povrchoch. (3) Klasifikujeme kvantové fázové prechody na troch vybratých spinových modeloch umiestnených na rôznych typoch hyperbolických mriežok vrátane Betheho mriežky.

Kľúčové slová:
fázové prechody, klasické a kvantové spinové mriežkové modely, hyperbolická geometria mriežok so zápornou krivosťou, tenzorové siete, tenzorové súčinové stavy, renormalizácia maticou hustoty, strednopoľová trieda univerzality

Abstract

The investigation of the behaviour of both classical and quantum systems on non-Euclidean surfaces near the phase transition point represents an interesting research area of the modern physics. However, due to the specific nature of the hyperbolic geometry, there have been no analytical solutions available so far and the potential of analytic and standard numerical methods is strongly limited. The task of finding an appropriate approach to analyze the fermionic models on hyperbolic lattices in the thermodynamic limit still remains an open question. In case of classical spin systems, a generalization of the Corner Transfer Matrix Renormalization Group algorithm has been developed and successfully applied to spin models on infinitely many regular hyperbolic lattices. In this work, we extend these studies to specific types of lattices. We also conclude that the hyperbolic geometry induces mean-field behaviour of all spin models at phase transitions. It is important to mention that no suitable algorithms for numerical analysis of ground-states of quantum systems in similar conditions have been implemented yet. In this thesis we offer a particular solution of the problem by proposing a variational numerical algorithm Tensor Product Variational Formulation, which assumes a quantum ground-state written in the form of a low-dimensional uniform tensor product state. We apply the Tensor Product Variational Formulation to three typical quantum models on a variety of regular hyperbolic lattices. Again, as in the case of classical spin systems, we conjecture the identical adherence to the mean-field-like universality class irrespective of the original model. The main outcomes of this thesis are the following: (1) We propose an algorithm for calculation and classification of the thermodynamic properties of the Ising model on triangular-tiled hyperbolic lattices. In addition, we investigate the origin of the mean-field universality on a series of weakly curved lattices. (2) We develop the Tensor Product Variational Formulation algorithm for the numerical analysis of the ground-state of the quantum systems on the hyperbolic lattices. (3) We study quantum phase transition phenomena for the three selected spin models on various types of the hyperbolic lattices including the Bethe lattice.

Keywords:
Phase Transition Phenomena, Classical and Quantum Spin Lattice Models, Hyperbolic Lattice Geometry, Tensor Networks, Tensor Product States, Density Matrix Renormalization, Mean-field Universality Classification

1. Introduction

The properties of both classical and quantum systems on non-Euclidean surfaces have been attracting researchers in various fields of modern physics. For example, experiments were performed with soft materials on conical geometry [1] and magnetic nanostructures on various negatively curved surfaces [2, 3, 4]. In addition, the influence of non-flatness of the underlying surface on the thermal properties of the system can be important in specific applications.

The main motivation of this PhD work is to investigate ground-state properties around phase transitions of strongly correlated systems, which are represented by a variety of Hamiltonians known in Solid-State Physics, when applied to negatively curved lattice geometries, often referred to as the so-called anti-de Sitter (AdS) space of the General Theory of Relativity. Here, wave functions of many-body interacting systems are intended to describe a non-trivial curved space, where time is excluded from consideration for the time being. The mutual relations among Solid-State Physics, General Theory of Relativity, and the Conformal Field Theory (CFT) enrich the interdisciplinary research, such as AdS-CFT correspondence known from the theory of Quantum Gravity [5, 6, 7, 8].

In order to accomplish such a nontrivial task, the physical space can be considered to be discrete. The entire discrete space is occupied by interacting multi-state spin variables with the distances as small as the Plank length (10^{-35}m) thus forming *a spin network*. The first elementary steps to tackle the given problem of the Quantum Gravity are studied in this thesis. In particular, we analyze relations between Gaussian curvature and correlations of the interacting spin particles. The off-criticality represented by non-diverging correlation length at phase transition is one of the key features to understand the negatively curved (AdS) geometry. The final step of this thesis will be the determination of a relation between the entanglement von Neumann entropy and the Gaussian curvature, which are crucial issues for the *holographic principle* in Quantum Gravity. Therefore, we have chosen quantum Heisenberg, XY, and transverse-field Ising models as the reference spin systems. Our intention is to confirm a concept of the holographic entanglement entropy [9, 10, 11]. It means that a non-gravitational theory is expected to live on the boundary of a subsystem $\partial\mathcal{A}$ of $(d+1)$-dimensional hyperbolic spaces. The entanglement entropy $S_{\mathcal{A}}$, associated with a reduced density matrix of $\mathcal{A}$, is a measure of the amount of information for the AdS/CFT correspondence. The entropy $S_{\mathcal{A}}$ is then related to a surface region $\partial\mathcal{A}$ in the AdS space. There is a duality in $(d+1)$-dimensional AdS and the d-dimensional system $\mathcal{A}$ in CFT.

In this thesis we begin with the study of simple spin models on regular hyperbolic lattices constructed by tessellation of congruent p-sided polygons with coordination number q, which are denoted as (p,q). The hyperbolic (p,q) lattices satisfy the condition $(p-2)(q-2) > 4$, exhibit constant negative curvature and their Hausdorff dimension is infinite if the thermodynamic limit is considered. On hyperbolic lattices the number of lattice sites N grows exponentially as the lattice diameter increases linearly. Also, the boundary effects are not negligible in the thermodynamic limit $N \to \infty$ on the hyperbolic lattices and, therefore, the spin systems exhibit phase transitions exclusively in the centre of the infinite hyperbolic lattice. Due to these specific conditions, the standard numerical tools developed for either classical or quantum systems (such as, Monte Carlo simulations, exact diagonalization, the coordinate Bethe Ansatz, the algebraic Bethe Ansatz or the vertex operator approach) face significant difficulties when applied to study phase transitions on hyperbolic lattices in the thermodynamic limit.

In case of the classical spin systems, the modified Corner transfer matrix renormalization group (CTMRG) algorithm was applied to an infinite series of hyperbolic $(p,4)$ lattices [12, 13, 14, 15]. Developing the original idea, we reformulate the CTMRG algorithm for use on the triangular $(3,q)$ as well as on weakly curved hyperbolic lattices, which are meant to be the missing complementary studies.

So far, an analogous algorithm designed for the ground-state analysis of quantum systems on hyperbolic surfaces has also been demanding. We expand a variational method, Tensor product variational formulation (TPVF) [16, 17] in order to find out an effective solution of the problem. Here, the quantum ground-state is approximated in the form of the tensor product state, which allows us to implement a generalization of the original CTMRG algorithm.

Our analyses of both the classical and the quantum spin systems confirm that the hyperbolic geometry causes that the mean-field universality behaviour at the phase transition point occurs, irrespective of the spin model used. We attribute this feature to the infinite Hausdorff dimension of the hyperbolic surfaces. Another key outcome of this work is an indirect analysis of the quantum spin models on the Bethe lattice, where the coordination number is fixed to be four. The Bethe lattice is attributed to the asymptotics of the $(p,4)$ lattices, where $p \to \infty$. These interesting outcomes have been published in refs. 18, 19, 17.

The results of our numerical analyses are briefly summarized in the following three sections. First, in section 2 we investigate phase transitions of the classical Ising model on the triangular $(3,q)$ lattice and weakly curved hyperbolic lattices. Next, the main idea of the TPVF algorithm is described in section 3. Finally, in section 4 we make use of the TPVF to study the quantum phase transition in the transverse-field Ising, XY and modified Heisenberg models on the series of the hyperbolic $(p,4)$ and $(4,q)$ lattices. We estimate the properties of the respective quantum models on the Bethe lattice.

2. Classical Ising model on $(3,q)$ hyperbolic lattices

The standard numerical methods for studying of classical lattice spin models are not easily applied under hyperbolic geometries. It has been shown that the CTMRG method is an appropriate candidate [12] when the $(p,4)$ lattice is considered under the condition $p \geq 4$. Applying a modification of the CTMRG algorithm, we investigated the thermodynamic behaviour of the classical Ising model on the complementary series of $(3,q)$ lattices constructed by tessellation of triangles. We assume $q \geq 6$, where $q = 6$ represents the Euclidean triangular lattice and $q > 6$ corresponds to the hyperbolic lattices.

We consider the classical ferromagnetic Ising model with Hamiltonian

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i, \tag{1}$$

where the spin variables σ_i are located on the vertices of the $(3,q)$ lattices, the symbol $\sum_{\langle i,j \rangle}$ denotes summation over pairs of the nearest-neighbouring spins, N stands for total number of spins in the system, $J > 0$ is a coupling constant and h is the external magnetic field. We are interested in the thermodynamic limit $N \to \infty$. Without loss of generality, we set the coupling constant J and the Boltzmann constant k_B to unity, and all thermodynamic functions are evaluated in dimensionless units.

We investigate the model via the modified CTMRG method, which reflects the specific geometry of the triangular $(3,q)$ lattices. The Boltzmann weight tensor $\mathbf{W}_B$ is assigned to a rhombus constructed from a pair of adjacent triangles and it is also necessary to introduce two different kinds of transfer tensors — the left tensor $\mathbf{L}_q$ and the right tensor $\mathbf{R}_q$. The expansion recurrent formulae of the corner transfer tensor $\mathbf{C}_q$ and the tensors $\mathbf{L}_q$, $\mathbf{R}_q$ in the iteration k in the simplified notation take the form

$$\begin{aligned}
\tilde{\mathbf{L}}_q^{(k+1)} &= \mathbf{W}_B \mathbf{L}_q^{(k)} \left(\mathbf{C}_q^{(k)}\right)^{q-6}, && (2)\\
\tilde{\mathbf{R}}_q^{(k+1)} &= \mathbf{W}_B \mathbf{R}_q^{(k)} \left(\mathbf{C}_q^{(k)}\right)^{q-6}, && (3)\\
\tilde{\mathbf{C}}_q^{(k+1)} &= \mathbf{W}_B \mathbf{L}_q^{(k)} \mathbf{R}_q^{(k)} \left(\mathbf{C}_q^{(k)}\right)^{2q-11}. && (4)
\end{aligned}$$

The boundary effects are not negligible on the hyperbolic lattices in the thermodynamic limit. In order to suppress the influence of the system boundary on the thermodynamic properties for the phase transition analysis in case of the hyperbolic lattices, we concentrate on the bulk properties of a sufficiently large inner region of the lattice [20, 12]. The local magnetization $M(h,T) \equiv \langle \sigma_\ell \rangle$ of a spin on

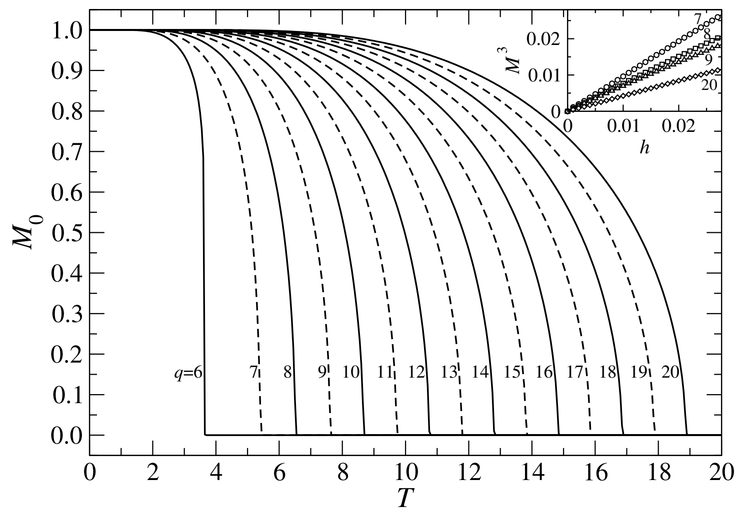

Figure 1: Spontaneous magnetizations $M_0(T) = M(h = 0, T)$ with respect to temperature T for $6 \leq q \leq 20$. The full and the dashed curves, respectively, distinguish the even and odd values of q. The inset shows the linear behaviour of the cubic power of the induced magnetization $M^3(h, T = T_{\rm pt}^{(q)})$ with respect the magnetic field h around the transition temperatures $T_{\rm pt}^{(q)}$ for $q \geq 7$.

the central lattice position ℓ is an example. Figure 1 shows the temperature dependence of the spontaneous magnetization $M_0(T) = M(h = 0, T)$ for the hyperbolic $(3, q)$ lattices with coordination numbers $6 \leq q \leq 20$.

The estimated transition temperature on the Euclidean (3,6) lattice $T_{\rm pt}^{(6)} = 3.641$ is quite close to the exact value $4/\ln 3 \approx 3.64096$ [21]. The transition temperature $T_{\rm pt}^{(q\geq7)}$ exhibits asymptotically linear dependence on the coordination number q, where the linearity appears already around $q \gtrsim 8$. This agrees with the linear dependence $T_{\rm pt}(q) = qJ/k_B$ observed in the mean-field model [21]. The identical transition temperatures $T_{\rm pt}^{(q)}$ were also determined from the analysis of the internal energy, specific heat and and the von Neumann entanglement entropy.

If a small magnetic field h is applied at the transition temperature $T_{\rm pt}^{(q\geq7)}$, the cubed induced magnetization $M^3(h, T = T_{\rm pt}^{(q)})$ is always linear around $h = 0$, as shown in the inset of fig. 1. Thus, the model satisfies the scaling relation $M(h, T = T_{\rm pt}) \propto h^{1/\delta}$ with the mean-field exponent $\delta = 3$. Similarly, the spontaneous magnetization near the transition point $T_{\rm pt}^{(q\geq7)}$ obeys $M(h = 0, T) \propto (T_{\rm pt}^{(q)} - T)^{\beta}$ with the mean-field exponent $\beta = \frac{1}{2}$ on hyperbolic lattices with $q \geq 7$. We confirm $\beta = \frac{1}{8}$, which agrees with the two-dimensional Ising universality class, on the Euclidean $(3, 6)$ lattice only.

We also conjectured that the correlation function $g(\mathbf{r}_i, \mathbf{r}_j)$ always decays expo-

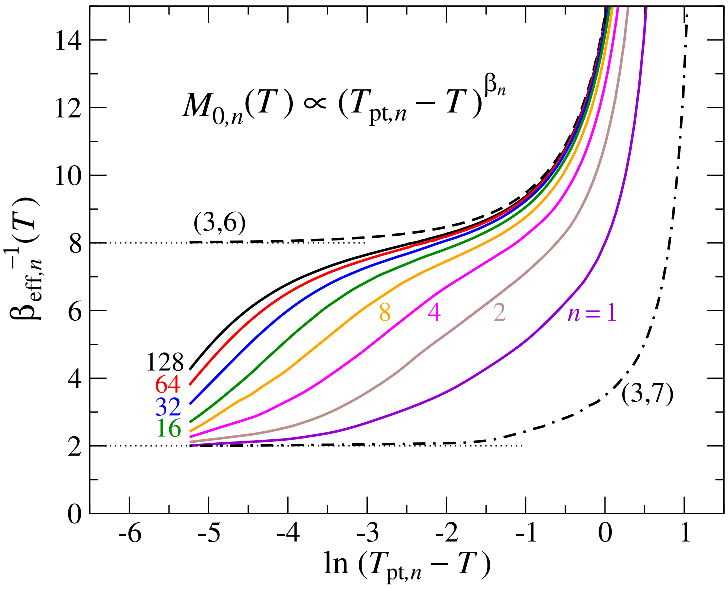

Figure 2: Inverse of the effective magnetic exponent $\beta_{\mathrm{eff},n}(T)$ as a function of the logarithmic distance from the transition temperature.

nentially on the hyperbolic $(3, q \geq 7)$ lattices regardless of the temperature, which is the direct consequence of the non-diverging correlation length ξ. The power law decay is observed only on the Euclidean $(3,6)$ lattice at the transition temperature $T_{\mathrm{pt}}^{(6)}$. Due to finite values of the correlation length even at the transition point, the term *critical* point on the hyperbolic lattices is not appropriate, since the critical point is always related to the divergence of the correlation length by definition. The mean-field exponents $\alpha = 0$, $\beta = \frac{1}{2}$, and $\delta = 3$, linear character of the dependence $T_{\mathrm{pt}}^{(q)}$, as well as the finiteness of the correlation length even at the transition temperature imply the mean-field nature of the classical Ising model on the hyperbolic surfaces.

In order to elucidate the origin of the mean-field universality induced by the hyperbolic geometry, we have investigated the Ising model on the slightly curved lattices constructed by distributing exceptional sites with coordination number seven throughout the originally flat Euclidean $(3,6)$ lattice. The exceptional sites form a regular pattern with a typical distance between nearest exceptional sites proportional to integer n. Varying the parameter n, the averaged curvature of the lattice $\kappa_n \propto -n^{-2}$ can be easily manipulated. In the limit $n \to \infty$ the flat $(3,6)$ lattice with $\kappa_\infty = 0$ is approached.

The magnetic exponent β_n in the scaling relation $M_{0,n}(T) \propto (T_{\mathrm{pt},n} - T)^{\beta_n}$ is determined through the effective exponent

$$\beta_{\mathrm{eff},n}(T) = \frac{\partial \ln M_{0,n}(T)}{\partial \ln\left(T_{\mathrm{pt},n} - T\right)}, \tag{5}$$

in the limit $T \to T_{\mathrm{pt},n}$ within the ferromagnetic ordered phase $T \leq T_{\mathrm{pt},n}$. Figure 2 shows $\beta_{\mathrm{eff},n}(T)$ obtained by means of the numerical derivative. When $T_{\mathrm{pt},n} - T \gtrsim 0.1$, the effective critical exponents $\beta_{\mathrm{eff},n}(T)$ for finite n follow the respective curves in the Euclidean case, however, they progressively bend to the mean-field value $\beta = 1/2$ as the transition temperature $T_{\mathrm{pt},n}$ is approached. A similar behaviour is observed also in case of the effective critical exponent δ_{eff} [19]. We also observed that the curves of the spontaneous magnetization $M_{0,n}(T)$ and the specific heat $C_{h,n}(h = 0, T)$ for the hyperbolic lattices with finite n continuously approach the curves for the Euclidean $(3, 6)$ lattice as $n \to \infty$.

Assuming the previous studies [12, 13, 14, 15] including the results presented here, we conclude that classical spin systems on any hyperbolic lattice belong to the mean-field universality class. The mean-field-like behaviour observed in the hyperbolic geometry originates in the infinite Hausdorff dimension of the hyperbolic lattices which obviously exceeds the critical value $d_c = 4$ [21, 22]. Notice that the CTMRG method does not affect the critical behaviour, since it accurately reproduces all of the critical exponents on the 2D Euclidean lattices, as has been shown in refs. 12, 13.

3. Tensor product variational formulation

Many analytical and computational techniques have been developed to study quantum spin models on the two-dimensional *Euclidean* lattices. However, the task of finding an appropriate approach to analyze quantum models on *hyperbolic* lattices still remains an open question. Here we introduce a novel and sufficiently accurate numerical algorithm called *Tensor product variational formulation* (TPVF) [16], which combines an ansatz for the quantum ground-state in the form of the Tensor product state (TPS) [23] with the Corner transfer matrix renormalization group scheme. This algorithm can be used to study ground-state of the quantum spin systems on the regular hyperbolic (p, q) lattices of constant negative Gaussian curvature in the thermodynamic limit, i.e., the number of the lattice vertices, where the spin variables are located, is infinite.

We assume that the Hamiltonian $\mathcal{H}$ of the model can be expressed as a sum of local Hamiltonians $G_k^{(p)}$ of the p-sided polygonal shape, in particular,

$$\mathcal{H} = \sum_{\langle k\rangle_p} G_k^{(p)}, \tag{6}$$

where k labels the polygons and the sum runs over the set of all indices of the lattice polygons $\langle k\rangle_p$. Our objective is to calculate an approximate ground-state of the system in the thermodynamic limit in the product form

$$|\Psi_p\rangle = \lim_{N\to\infty} \sum_{\sigma_1\sigma_2\cdots\sigma_N} \prod_{\langle k\rangle_p} W_p(\{\sigma_k\})|\sigma_1\sigma_2\cdots\sigma_N\rangle, \tag{7}$$

where N stands for the total number of the lattice spins and σ_j, $j = 1, ..., N$, marks one of the two base states $\downarrow$ or $\uparrow$ of the j^{th} lattice spin. The summation runs over the 2^N base spin states $|\sigma_1\sigma_2\cdots\sigma_N\rangle$, and $W_p(\{\sigma_k\})$ are the elements of the p-rank tensor W_p depending on p spins $\sigma_{k_1}, \ldots, \sigma_{k_p}$ on the k^{th} lattice polygon. The symbol $\{\sigma_k\}$ stands for one of the 2^p base configurations of a multi-spin variable representing the group of spins $\sigma_{k_1}, \ldots, \sigma_{k_p}$. All the tensors W_p are considered to be identical, therefore, the set of 2^p tensor elements $W_p(\{\sigma\})$, where the subscript k has been omitted due to the uniformity of the tensors W_p, uniquely describes the state $|\Psi_p\rangle$, i.e. $|\Psi_p\rangle = |\Psi_p[W_p(\{\sigma\})]\rangle$.

We regard $|\Psi_p^*\rangle$ as the best approximation of the ground-state within the class of TPS $|\Psi_p\rangle$, if the minimum of the energy normalized per bond,

$$E_0^{(p)} \equiv \min_{\Psi_p} \lim_{N_b\to\infty} \frac{1}{N_b} \frac{\langle\Psi_p|\mathcal{H}|\Psi_p\rangle}{\langle\Psi_p|\Psi_p\rangle}, \tag{8}$$

is obtained for $|\Psi_p^*\rangle$. Here, N_b denotes the total number of bonds in the system. The energy $E_0^{(p)}$, due to its variational origin, serves as an upper bound of the true ground-state energy per bond $\mathcal{E}_0^{(p)}$.

Since the structure of every local Hamiltonian $G_k^{(p)}$ does not depend on k (we investigate the system in the thermodynamic limit), the variational problem in (8) is equivalent to minimization of the local energy per bond of an arbitrary polygon in the lattice center (in order to avoid boundary effects)

$$E_0^{(p)} = \frac{2}{p} \min_{\Psi_p} \frac{\langle \Psi_p | G_\ell^{(p)} | \Psi_p \rangle}{\langle \Psi_p | \Psi_p \rangle}\,, \tag{9}$$

where ℓ is the index of the selected polygon and the normalization factor $2/p$ reflects the fact that the p bonds of each polygon are shared with its neighbors. Moreover, if we utilize the tensor product structure of the state $|\Psi_p\rangle$, we can express the denominator $\langle \Psi_p | \Psi_p \rangle \equiv \mathcal{D}(W_p(\{\sigma\}))$ and the numerator $\langle \Psi_p | G_\ell^{(p)} | \Psi_p \rangle \equiv \mathcal{N}(W_p(\{\sigma\}))$ as functions of the tensor elements $W_p(\{\sigma\})$ only. Consequently, our variational problem transforms onto a multi-dimensional minimization over 2^p tensor elements $W_p(\{\sigma\})$

$$E_0^{(p)} = \frac{2}{p} \min_{W_p(\{\sigma\})} \frac{\mathcal{N}(W_p(\{\sigma\}))}{\mathcal{D}(W_p(\{\sigma\}))}\,. \tag{10}$$

The set of the 2^p variational parameters (tensor elements) $W_p(\{\sigma\})$ is further significantly reduced by utilizing the symmetries of the local Hamiltonian $G_\ell^{(p)}$. For example, rotational and spin-ordering symmetries are present in the transverse-field Ising model and, in addition, the spin-inversion symmetry appears if there is no preferred spin alignment in the system (such as in the XY model, the Heisenberg model, as well as in the TFIM at and above the phase transition magnetic field).

The Tensor Product Variational Formulation algorithm consists of two parts. The first one evaluates the ratio in (10) by applying the CTMRG method separately to the numerator and the denominator for a given set of the variational parameters $W_p(\{\sigma\})$. The second part contains a multi-dimensional minimizer, the Nelder-Mead simplex algorithm [24,25,26], which uses the first part to search for the optimized set of the variational parameters $W_p^*(\{\sigma\})$, which minimize the ratio in (10).

The TPS approximation (7) of the quantum ground-state may be a limiting factor regarding the accuracy of the TPVF on the Euclidean lattices near the critical point, where the correlation length ξ diverges. The reason is the low dimension of the tensors W_4 in the TPS approximation which suppresses the quantum long-range correlations on the Euclidean $(4,4)$ lattice near the criticality. As a result, the

TPS approximation (7) induces mean-field-like behaviour near the quantum phase transition irrespective of the true universality class the original model belongs in.

On the contrary, it is expected that all the classical and quantum spin lattice models on various types of the hyperbolic surfaces belong to the mean-field universality class, since the Hausdorff dimension of the hyperbolic lattices is infinite, which evidently exceeds the critical values $d_C = 4$ and $d_C = 3$, respectively. This was confirmed in studies [13, 12, 18] of classical spin models on the hyperbolic lattices, where the exponential decay of the density matrix spectra and the correlation function result in the non-critical phase transition, since the correlation length, $\xi \lesssim 1$, is always finite, reaching its maximal value at the phase transition [27]. We assume a similar scenario also in case of the quantum systems. Hence, the mean-field approximation of the TPVF algorithm induced by low dimensions of TPS (7) is not in conflict with the mean-field-like behaviour of quantum models on the hyperbolic lattice geometry. For this reason, we conjecture the TPFV analysis of the models on the hyperbolic lattices is more accurate than on the Euclidean ones.

4. Quantum spin models on hyperbolic lattices

Applying the TPVF algorithm, we study the ground-state properties and the phase transition of the quantum transverse field Ising model (TFIM), XY, and modified Heisenberg models in the thermodynamic limit on a series of hyperbolic $(p, 4)$ lattices with the lattice parameter $p \in \{5, 6, \ldots, 11\}$. Apart from the set, we include two additional cases: $p = 4$ being the Euclidean square lattice and the asymptotic case $p \to \infty$, which is associated with the Bethe lattice. Next, an analogous study is performed on the complementary set of the $(4, q)$ lattices for $4 \leq q \leq 70$, where the transverse field Ising model is investigated.

The Hamiltonian of the three models with N spins is given by formula

$$\mathcal{H}(J_{xy}, J_z) = -\sum_{\langle i,j \rangle} \left[J_{xy} \left(\boldsymbol{\sigma}_i^x \boldsymbol{\sigma}_j^x + \boldsymbol{\sigma}_i^y \boldsymbol{\sigma}_j^y \right) + J_z \boldsymbol{\sigma}_i^z \boldsymbol{\sigma}_j^z \right] - h \sum_{i=1}^{N} \boldsymbol{\sigma}_i^x, \tag{11}$$

where $\boldsymbol{\sigma}^x$, $\boldsymbol{\sigma}^y$, $\boldsymbol{\sigma}^z$ are the spin operators given by the Pauli matrices in the z-representation. We consider the ferromagnetic Ising and the XY model with coupling constants $J_{xy} = 0$, $J_z = J$ and $J_{xy} = J$, $J_z = 0$, respectively. Without loss of generality, we set $J = 1 > 0$. The modified Heisenberg model is specified by the choice $J_{xy} = J = 1$, $J_z = -J = -1$. We study the XY and the modified Heisenberg models at zero magnetic field.

The TFIM undergoes a quantum phase transition at a nonzero magnetic field $h_t^{(p)} > 0$, which is characterized by a non-analytic behaviour in the expectation value of the spontaneous magnetization $\langle S_p^z \rangle$ when $\langle S_p^z \rangle$ approaches the zero value from the ordered phase ($h < h_t^{(p)}$). The spontaneous magnetization is evaluated as

$$\langle S_p^z \rangle = \frac{\langle \Psi_p^* | \boldsymbol{\sigma}_\ell^z | \Psi_p^* \rangle}{\langle \Psi_p^* | \Psi_p^* \rangle}, \tag{12}$$

where $|\Psi_p^*\rangle$ is the approximative ground state uniquely defined via (7) by inserting the optimal tensor elements $W_p^*(\{\sigma\})$ and ℓ labels an arbitrary spin in the central polygon of the lattice in order to suppress boundary effects. The resulting dependence of the magnetization $\langle S_p^z \rangle$ with respect to the magnetic field h near the phase transition field $h_t^{(p)}$ is plotted in the upper graph of figure 3.

The lower graph in fig 3 shows the squared transversal magnetization $\langle S_p^z \rangle^2$, where we point out the linearity of the squared magnetization if approaching the phase transition field $h_t^{(p)}$. Such a dependence confirms the mean-field exponent $\beta_p = \frac{1}{2}$ in the scaling relation $\langle S_p^z(h) \rangle \propto \left(h_t^{(p)} - h \right)^{\beta_p}$ regardless of the lattice parameter p. The incorrect mean-field-like behaviour near the phase transition on the Euclidean lattice represented by the mean-field value $\beta_4 = \frac{1}{2}$ is attributed to the

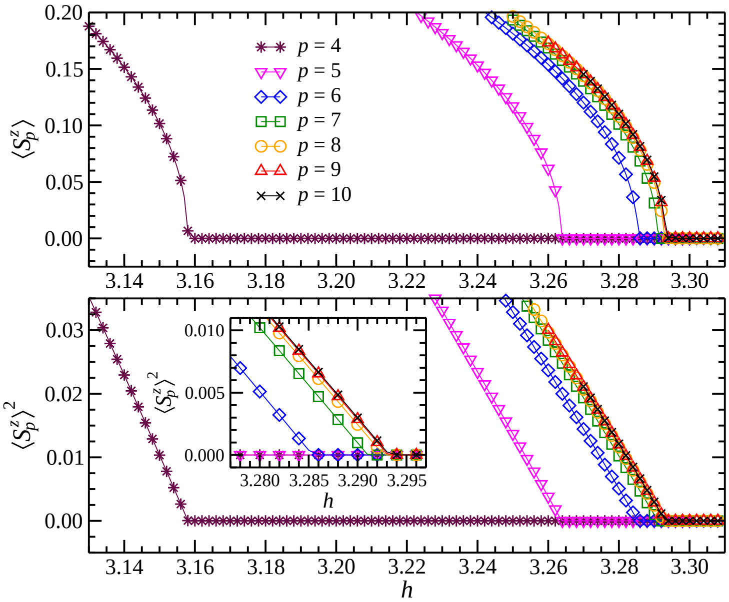

Figure 3: The spontaneous magnetization $\langle S_p^z \rangle$ (the upper graph) and its square $\langle S_p^z \rangle^2$ (the lower graph) in the vicinity of the phase transitions with respect to the magnetic field h for $p \in \{4, 5, \ldots, 10\}$. The inset shows the detailed zoomed-in behaviour for higher values of p.

exclusion of long-range correlations caused by the TPS approximation (7) which is built up by the tensors W_4 of the too low dimension. As a reference, the numerical TRG analysis [28] gives correct $\beta_4^{\rm TRG} = 0.3295$ on the Euclidean $(4,4)$ lattice, which is also in agreement with Monte Carlo simulations.

The phase transition fields $h_t^{(p)}$ calculated from the spontaneous magnetization are shown in fig. 4. The identical transition fields $h_t^{(p)}$ were determined also from analysis of the magnetic susceptibility and the optimal variational parameters $W_p^*(\{\sigma\})$. Applying an exponential fit to the data in the form $h_t^{(p)} = h_t^{(\infty)} + a_1 \exp(a_2 p)$, we calculated the asymptotic phase transition field of the TFIM on the Bethe lattice $h_t^{(\infty)} = 3.29332$. If compared to the most relevant value of the critical magnetic field for the TFIM on the Euclidean $(4,4)$ lattice by the TRG algorithm [28] we conjecture the relative error 3.7% of our result $h_t^{(4)}$.

The ground-state energies $E_0^{(p)}$ of the XY and the modified Heisenberg models have been studied in the absence of magnetic field on the series of the regular $(p,4)$ lattices with $4 \le p \le 11$. The resulting dependence of the ground-state energy per bond $E_0^{(p)}$ on the lattice parameter p differs considerably for the two models. While the energies $E_0^{(p)}$ of the XY model form a monotonically increasing and exponentially saturated sequence with increasing p, the modified Heisen-

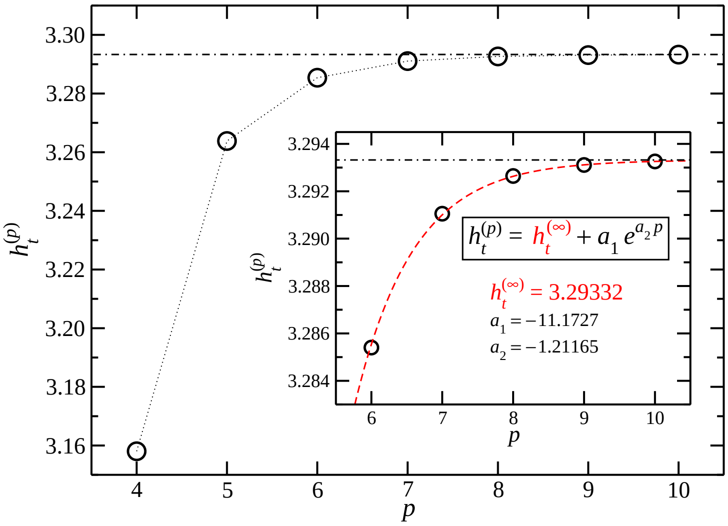

Figure 4: The phase transition field $h_t^{(p)}$ of the TFIM with respect to the lattice parameter p. The horizontal dot-dashed line represents the estimated asymptotic value $h_t^{(\infty)} = 3.29332$.

berg model induces a saw-like dependence containing the separated upper (odd p) and the lower (even p) branches, both of them converging exponentially fast to the common asymptotic value $E_0^{(\infty)}$ which corresponds to the ground-state energy on the Bethe lattice with the coordination number four. We conjecture that the ground-state energies per bond of the XY and the Heisenberg models, respectively, occur at $E_0^{(\infty)} = -1.08083446$ and -1.291944. The latter value is common also to the antiferromagnetic Heisenberg model. The saw-like pattern in case of the modified Heisenberg model may be attributed to the fact, that if p is even, the ground-state is antiferromagnetic, while if p is odd, the ferromagnetic state is obtained. We have not found any theoretical reasoning for the exponential convergence of the ground-state energies $E_0^{(p)}$ yet. The energies $E_0^{(4)}$ calculated by TPVF on the Euclidean $(4,4)$ lattice for both the XY and the Heisenberg models are higher if compared to the results of the Monte Carlo simulations [29, 30]. The respective relative errors are 1.2% and 2.2%.

The influence of the varying coordination number q on the ground-state properties of the quantum transverse field Ising model in the thermodynamic was investigated on the series of the $(4,q)$ lattices, where $4 \leq q \leq 70$. In our analysis, we focused on the von Neumann entropy of the system $S = -\mathrm{Tr}(\rho \log_2 \rho)$, where two different types of reduced density matrices ρ were chosen. No qualitative difference between the results in the two respective cases was observed.

The peak of the entropy curve, which marks the phase transition field $h_{\mathrm{t}}^{(4,q)}$, shifts to the right and its maximum decreases as q increases. We determined

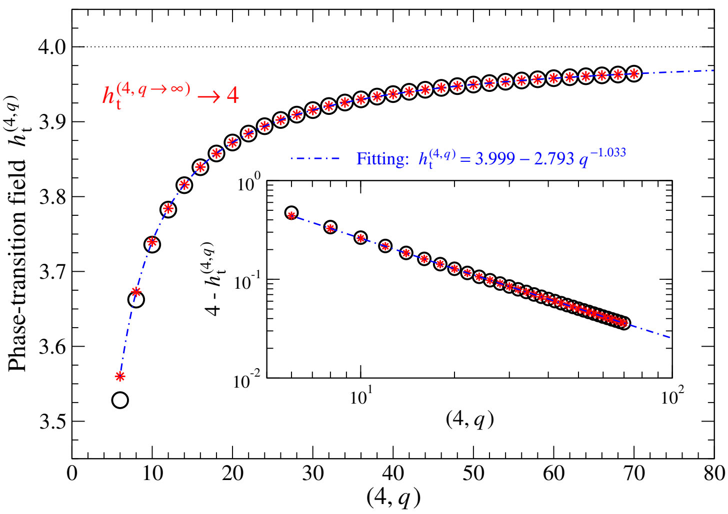

Figure 5: The phase transition field $h_t^{(4,q)}$ as a function of the coordination number q for $4 \leq q \leq 70$. The open circles mark the calculated data, while the blue dot-dashed line and the red stars illustrate the polynomial fitting function and the predicted values for the integers q, respectively. The inset illustrates the linearity of the curve $h_t^{(4,q\to\infty)} - h_t^{(4,q)} = 4 - h_t^{(4,q)}$ with respect to q in the log-log scale, which supports the polynomial fit.

the transition field $h_t^{(4,q)}$ as the magnetic field h which yielded the optimal value $S_{\mathrm{MAX}}(q) = \max_h S(h)$. A polynomial character of the dependency $S_{\mathrm{MAX}}(q)$ was observed. Figure 5 depicts the phase transition field $h_t^{(4,q)}$ as a function of the coordination number q. A detailed analysis, cf. the inset in fig. 5, suggests that the best description of the data can be obtained by a polynomial fitting function. The parameters of the optimal fitting function are shown in the graph. Note that, according to the fit, as q increases, the transition field $h_t^{(4,q)}$ tends to the asymptotic value $h_t^{(4,q\to\infty)} = 4$. The observed polynomial curve for $h_t^{(4,q)}$ represents a new feature if compared to the results of the classical Ising model on the (p,q) lattices, where a linear dependence of the transition temperature $T_{\mathrm{pt}}^{(q)} \propto q$ for large q was detected [31]. The origin of the polynomial behaviour of both the $h_t^{(4,q)}$ and $S_{\mathrm{MAX}}(q)$ has not been clarified yet and requires futher analyses.

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